

THE LINK SURGERY OF $S^2 \times S^2$ AND SCHARLEMANN'S MANIFOLDS

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Abstract

Fintushel-Stern's knot surgery gave many pairs of exotic manifolds, which are homeomorphic but non-diffeomorphic. We show that if an elliptic fibration has two parallel, oppositely oriented vanishing circles (for example $S^2 \times S^2$ or Matsumoto's S^4), then the knot surgery gives rise to standard manifolds. The diffeomorphism can give an alternative proof that Scharlemann's manifold is standard (originally by Akbulut [Ak1]).

1 Introduction.

1.1 Knot surgery.

Let X be a 4-manifolds containing the cusp neighborhood C , which is the well-known elliptic fibration over D^2 with one cusp singularity. In [FS] R. Fintushel and R. Stern constructed exotic structures by performing the knot surgery of a general fiber of X near the cusp fiber. The knot surgery on X that containing C is defined as follows. Let K be a knot in S^3 . For a general fiber T of C the surgery

$$X_K := [X - \nu(T)] \cup_{\varphi_0} [(S^3 - \nu(K)) \times S^1]$$

is called (*Fintushel-Stern's*) *knot surgery*. Here $\nu(\cdot)$ stands for the interior of the tubular neighborhood. The gluing map

$$\varphi_0 : \partial\nu(K) \times S^1 \rightarrow \nu(T^2) = T^2 \times \partial D^2$$

satisfies the following.

$$\text{the meridian of } K \times \{\text{pt}\}, \{\text{pt}\} \times S^1 \rightarrow \alpha, \beta,$$

$$\text{the longitude of } K \times \{\text{pt}\} \rightarrow \{\text{pt}\} \times \partial D^2$$

where α, β are generators of $H_1(T)$.

We can easily check that X_K is homeomorphic to X from Freedman's celebrated result if X is simply connected and closed. When is (X, X_K) an exotic pair? Fintushel and Stern proved the following formula on the Seiberg-Witten invariant.

$$SW_{X_K} = SW_X \cdot \Delta_K, \tag{1}$$

where Δ_K is the Alexander polynomial of K . This formula implies that many knot-surgeries change the differential structures. However in the case where $\Delta_K(t) = 1$ or $SW_X = 0$, it is in general unknown whether the pair is exotic or not.

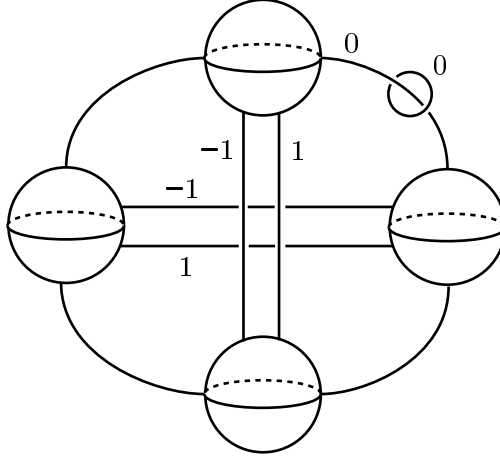


Figure 1: The cusp neighborhood and anti-cusp neighborhood in $S^2 \times S^2$.

On the other hand it is well-known that $S^2 \times S^2$ admits achiral Lefschetz fibration containing C , namely $S^2 \times S^2$ is diffeomorphic to the double $\overline{C} \cup C$ of C . The diagram is drawn in Figure 1.

We denote $\overline{C} \cup C_K$ by A_K . Since $SW_{S^2 \times S^2} = 0$ holds, SW-invariant cannot distinguish whether A_K is exotic or not.

In [Ak2] S.Akbulut showed that A_{3_1} is diffeomorphic to $S^2 \times S^2$. The diffeomorphism is due to his result [Ak1]. This says the existence of exotic embedding of C into $S^2 \times S^2$. In the article we will show the following.

Theorem 1. *Let K be any knot. Then A_K is diffeomorphic to $S^2 \times S^2$.*

1.2 Link surgery.

Let L be a link in S^3 . We define A_L to be the link surgery of $S^2 \times S^2$ along a general fiber near the cusp fiber. The precise definition of link surgery is in Section 4. Then we give a classification of A_L by applying the same method as the knot case.

Theorem 2. *Let $L = K_1 \cup \dots \cup K_n$ be any n -component link. Then A_L is diffeomorphic to*

$$A_L = \begin{cases} \#^{2n-1} S^2 \times S^2 & \sum_{i \neq j} lk(K_i, K_j) = 0 \text{ (2) for any } i \\ \#^{2n-1} \mathbb{C}P^2 \#^{2n-1} \overline{\mathbb{C}P^2} & \text{otherwise.} \end{cases}$$

In Section 4 we prove the theorem.

1.3 Scharlemann's manifolds.

We define closed 4-manifolds $B_K^\epsilon(\gamma)$ ($\epsilon = 0, 1$) to be the surgery:

$$[S_{-1}^3(K) \times S^1 - \nu(\gamma \times \{\text{pt}\})] \cup_{\phi'} S^2 \times D^2,$$

where γ is a knot in $S_{-1}^3(K)$. The diffeomorphism type depends only on free homotopy class of the map $S^1 \hookrightarrow S_{-1}^3(K)$.

The map ϕ' is gluing map $S^2 \times \partial D^2 \rightarrow \partial D^3 \times S^1 = \partial \nu(\gamma \times \{\text{pt}\})$ where $\phi'(x, t) = (\phi(x, t), t)$. If $\epsilon = 0$, then the gluing map is trivial ($\Leftrightarrow \phi(x, \cdot) = 0 \in \pi_1(SO(3))$) and if $\epsilon = 1$, then the gluing map is non-trivial ($\Leftrightarrow \phi(x, \cdot) \neq 0$). In this paper

$B_K^1(\gamma)$ is called *Scharlemann's manifold*. Any Scharlemann's manifold is homotopy equivalent to $S^3 \times S^1 \# S^2 \times S^2$ (see [Sc]). More strongly $B_K^\epsilon(\gamma)$ is homeomorphic to $S^3 \times S^1 \# S^2 \times S^2$ by Freedman's result. It had been for a long time unknown whether $B_{3_1}^1(\gamma)$ is diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$ or not. Akbulut [Ak1] showed the following.

Theorem 3 ([Ak1]). *Let γ_0 be the meridian of $S^3 - 3_1 \subset S_{-1}^3(3_1)$. $B_{3_1}^1(\gamma_0)$ is diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$.*

Here we state the following as the third main theorem.

Theorem 4. *Suppose that K is any knot. Let γ_0 be the meridian of $S^3 - K \subset S_{-1}^3(K)$. Then $B_K^1(\gamma_0)$ is diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$.*

Moreover in Section 5.2 we will consider the diffeomorphism type of $B_{3_1}^\epsilon(\gamma)$ for other homotopy class γ .

After I posted the article on arXiv, S. Akbulut also proved the same results in [Ak4] in a simple way. He uses the fact that $S^2 \times S^2$ is the double of a cusp neighborhood C , however my proof is available whenever there exist two, parallel, opposite, fishtail singular fiber in an elliptic fibration.

Acknowledgments.

The author originally was taught the candidates A_K of exotic $S^2 \times S^2$ by Professor M. Akaho ([Aka]). This paper is the negative but complete answer for his question. I thank him for telling me about attractive 4-dimensional world.

Last year we posed the diffeomorphism for figure-8 surgery of $S^2 \times S^2$, however the proof included a wrong part. The author deeply apologizes for Differential Topology Seminar at Kyoto University at 2009 because the author explained the wrong proof. The error was pointed out by Professor S. Akbulut at Michigan state University in 2010, so the author expresses the gratitude for him.

The author thanks for Professor Masaaki Ue, Dr. Kouichi Yasui and Shohei Yamada for giving me many useful comments by some seminars. The research is partially supported by JSPS Research Fellowships for Young Scientists (21-1458).

2 The logarithmic transformation.

2.1 Definition and Gompf's result.

In this section we shall define the logarithmic transformation. Let X be an oriented 4-manifold and $T \subset X$ a embedded torus with self-intersection 0. The surgery

$$X_{T,p,q,\gamma} = [X - \nu(T)] \cup_\varphi D^2 \times T^2,$$

is called *logarithmic transformation*, where the gluing map $\varphi : \partial D^2 \times T^2 \rightarrow \partial \nu(T)$ satisfies $\varphi(\partial D^2 \times \{\text{pt}\}) = q(\{\text{pt}\} \times \gamma) + p(\partial D^2 \times \{\text{pt}\})$. It is well-known that the diffeomorphism type of logarithmic transformation depends only on the data (T, p, q, γ) . The integer p is *multiplicity* of the logarithmic transformation, γ is the *direction* and q is *auxiliary multiplicity*.

If $p = 1$, then we call $X_{T,1,q,\gamma}$ a q -fold Dehn twist of $\partial \nu(T)$ along T parallel to γ .

Lemma 1 (Lemma 2.2 in [G]). *Suppose $N = D^2 \times S^1 \times S^1$ is embedded in a 4-manifold X . Suppose there is a disk $D \subset X$ intersecting N precisely in $\partial D = \{q\} \times S^1$ for some $q \in \partial D^2 \times S^1$, and that the normal framing of D in X differs from the product*

framing on $\partial D \subset \partial N$ by ± 1 twist. Then the diffeomorphism type of X does not change if we remove N and reglue it by a k -fold Dehn twist of ∂N along $S^1 \times S^1$ parallel to $\gamma = \{q\} \times S^1$.

The manifold $F := N \cup \nu(D)$ in Lemma 1 is called *fishtail neighborhood*, and the diagram is Figure 2.

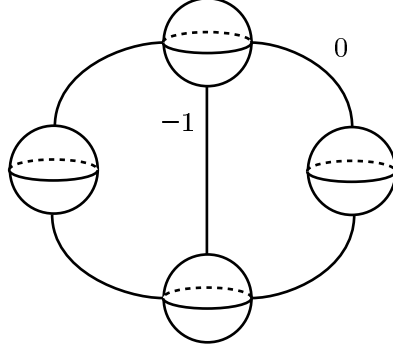


Figure 2: Fishtail neighborhood.

2.2 One strand twist.

Let K_1 be any knot in S^3 and K_2 the meridian of K_1 . We consider the knot surgery X_{K_1} whose attaching map is φ_0 , where a 4-manifold X contains C . The subset $T_2 := K_2 \times S^1 \subset X_{K_1}$ is a self-intersection 0 torus. Thus the neighborhood $N_2 = \nu(K_2) \times S^1$ is the trivial normal bundle over T_2 . Since any parallel copy $K'_2 \subset \partial N_2$ of K_2 by the obvious trivialization of N_2 is isotopic to one of vanishing circle of C_{K_1} , there exists a disk $D \subset C_{K_1}$ with $\partial D = K'_2$. Thus the framing of ∂D coming from the trivialization of $\nu(D)$ differs from the normal framing of the trivialization of N_2 by -1 . As a result $N \cup \nu(D)$ is the fishtail neighborhood.

From Lemma 1 n -fold Dehn twist of ∂N_2 along T_2 parallel to K_2 does not change the diffeomorphism type. For any integer n we define φ_n to be a diffeomorphism satisfying

$$\varphi_n^{-1}(\{\text{pt}\} \times \partial D^2) = [\text{longitude of } K] + n[\text{meridian of } K], \quad (2)$$

where other images $\varphi(\text{meridian of } K)$ and $\varphi(\{\text{pt}\} \times S^1)$ are the same as φ_0 . We define $X_{K,n}$ to be

$$X_{K,n} := [X - \nu(T)] \cup_{\varphi_n} [(S^3 - \nu(K)) \times S^1]$$

The previous paragraph means

$$X_{K_1,n} \cong X_{K_1,0} = X_{K_1}.$$

The diffeomorphism can be also understood by handle calculus as Figure 3. This move is by Akbulut's method in [Ak1]. The left in Figure 3 is the 4_1 surgery of the cusp neighborhood. Sliding the top -1 framed 2-handle over one of two 0 framed 2-handles below, we get the right-top one in Figure 3. Sliding upper 0 framed 2-handle over the -1 framed 2-handle, we have the right-bottom picture. This process increases the framing of the knot by 1. Iterating the process or the inverse one, we can change the framing to the arbitrary integer.

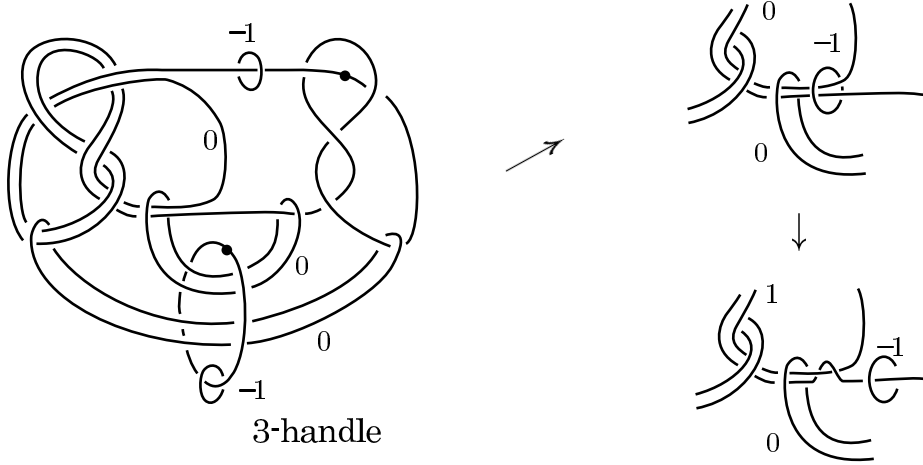


Figure 3: The change of framing.

3 The knot surgery of $S^2 \times S^2$.

Finding a hidden fishtail neighborhood in $\overline{C} \cup C_K$, we shall prove diffeomorphisms.

Proof of Theorem 1.

3.1 Three strand twist.

Let L be a two component link as in Figure 4. The box is some tangle which presents

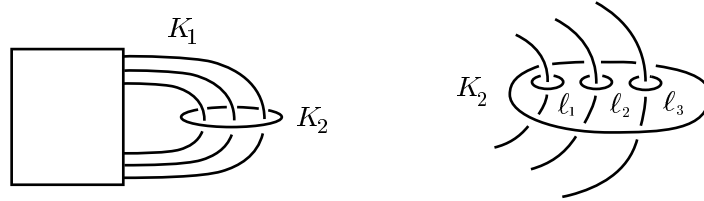


Figure 4: $L = K_1 \cup K_2$ and ℓ_1, ℓ_2, ℓ_3 .

K_1 . We consider the knot surgery $\overline{C} \cup C_{K_1}$ where φ_0 is the gluing map of the knot surgery. The torus $T_2 = K_2 \times S^1 \subset [S^3 - \nu(K_1)] \times S^1$ has the trivial neighborhood in A_{K_1} . We denote the neighborhood of the torus by N_2 .

Our aim here is to construct a fishtail neighborhood in which $K_2 \times S^1$ is a general fiber. K_2 is clearly homologous to the union of meridians ℓ_1, ℓ_2 and ℓ_3 of K_1 via an obvious 3-punctured disk P as in Figure 4. Here any ℓ_i lies in the boundary of N_1 which is the neighborhood of K_1 . Each image $\varphi_0(\ell_i)$ is parallel to two vanishing circles in $\overline{C} \cup C_{K_1}$ as in Figure 5. Here the twisted 1-framed 2-handle is obtained by sliding a trivial 0-framed 2-handle (one of 2-, 3-handle canceling pair) to the 1-framed 2-handle.

Here we will construct three annuli A_1, A_2 and A_3 that each side of ∂A_i is $\varphi_0(\ell_i)$. A_1 is as in Figure 6 and the right side of ∂A_1 is $\varphi_0(\ell_1)$. A_2 and A_3 are as in the left and right in Figure 7 respectively. A_3 runs through the digged 2-handle (the dotted 1-handle) once. In addition the right sides of ∂A_2 and ∂A_3 are $\varphi_0(\ell_2)$ and $\varphi_0(\ell_3)$. Obviously A_1, A_2 and A_3 are disjoint annuli in $\overline{C} \cup C_{K_1}$.

Another sides of ∂A_i are the boundaries of 2-disks parallel to the cores of the 2-handles in Figure 5. Capping the three 2-disks C_1 , C_2 and C_3 to three components of $\partial(P \cup A_1 \cup A_2 \cup A_3) - K_2$, we obtain an embedded disk $D := P \cup A_1 \cup A_2 \cup A_3 \cup C_1 \cup C_2 \cup C_3$ in $\overline{C} \cup C_{K_1}$ whose boundary is K_2 .

The framing in $\partial\nu(D)$ coming from the trivialization of $\nu(D)$ differs from the framing of K_2 coming from the normal bundle of N_2 by $-1 + 1 + 1 = 1$. Therefore $N_2 \cup \nu(D)$ is diffeomorphic to \overline{F} . Changing the isotopies of $\varphi_0(\ell_i)$ to two -1 -framed 2-handles and one 1 -framed 2-handle, we can also embed F .

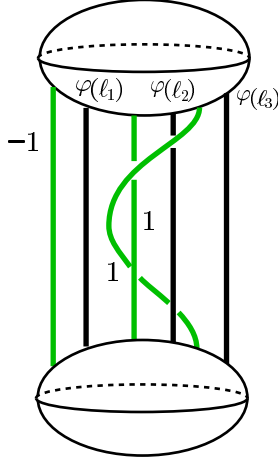


Figure 5: An isotopy of $\varphi(\ell_i)$.

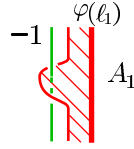


Figure 6: A_1 .

Hence ± 1 -fold Dehn twist along T_2 parallel to K_2 gives rise to the same manifolds A_{K_1} and $\overline{C} \cup C_{K_3, n}$. The integer n is one of $\mp 1, \mp 9$. Here K_3 is the knot obtained by the ± 1 -Dehn surgery along K_2 as in Figure 8. By one strand twist in Section 2.2 we have $A_{K_3} \cong \overline{C} \cup C_{K_3, n} \cong A_{K_1}$.

If we replace three strand twist with odd strand twist, we can construct diffeomorphisms.

3.2 Ohyama's unknotting operation.

Y. Ohyama in [Oh] has proven that local three strand twist is an unknotting operation of knots. Therefore for any knot K there exists a finite sequence of local three strand twists: $K = k_0 \rightarrow k_1 \rightarrow \cdots \rightarrow k_n = \text{unknot}$. The sequence implies the sequence of diffeomorphisms:

$$A_K = A_{k_0} \cong A_{k_1} \cong \cdots \cong A_{k_n} = S^2 \times S^2.$$

□

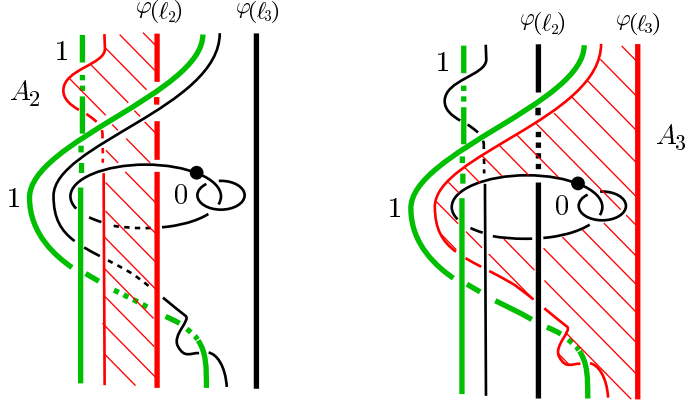


Figure 7: Two embedded annuli A_2, A_3 .

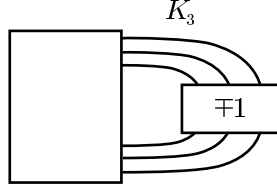


Figure 8: K_3 : a ± 1 full twist of K_1 . The right box is the ∓ 1 full twist.

Using the diffeomorphism, we obtain infinitely many embeddings:

$$C \hookrightarrow C \cup \overline{C_K} = S^2 \times S^2.$$

However whether the embeddings are mutually non-diffeomorphic is unknown.

Remark 1. The diffeomorphism by the three strand twist can be applied to any knot surgery of any genus-1 achiral Lefschetz fibration with two oppositely oriented fishtail (or cusp) fiber. For example S^4 admits genus-1 achiral Lefschetz fibration with two opposite fishtail fiber (Y. Matsumoto [M] and see Figure 8.38 in [GS] for the diagram). Any knot surgery of Matsumoto's Lefschetz fibration on S^4 gives rise to standard S^4 .

4 Link surgery case.

Let $L = K_1 \cup \cdots \cup K_n$ be an n -component link and X_i ($i = 1, \dots, n$) oriented 4-manifolds which contains the cusp neighborhood C_i . Let T_i be a general fiber of C_i . By the gluing map

$$\varphi_i : \partial\nu(K_i) \times S^1 \rightarrow \partial\nu(T_i) = T_i \times \partial D^2$$

satisfying

$$\begin{aligned} \varphi_i(l_i \times \{\text{pt}\}) &= \{\text{pt}\} \times \partial D^2 \\ \varphi_i(m_i \times \{\text{pt}\}) &= \alpha_i, \quad \varphi_i(\{\text{pt}\} \times S^1) = \beta_i, \end{aligned}$$

where l_i and m_i are the longitude and meridian of K_i and α_i, β_i are vanishing circles of $\partial\nu(T_i)$, we define an operation

$$\prod_{i=1}^n X_i \rightarrow (S^3 - \nu(L)) \times S^1 \cup_{\varphi_i} [X_i - \nu(T_i)].$$

We call the operation *link surgery* of (X_1, \dots, X_n) along L and denote it by $X(X_1, \dots, X_n; L)$. The Seiberg-Witten invariant of $X(X_1, \dots, X_n; L)$ is computed as follows:

$$SW_{X(X_1, \dots, X_n; L)} = \Delta_L(t_1, \dots, t_n) \cdot \prod_i^n SW_{E(1) \#_{T=T_i} X_i},$$

where $\Delta_L(t_1, \dots, t_n)$ is the n variable Alexander polynomial of L and $E(1) \#_{T=T_i} X_i$ is the fiber sum of the elliptic fibration $E(1)$ and X_i along general fibers T and T_i respectively. The definition of the fiber sum can be seen in [FS].

Here we consider the link surgery of $\prod_{i=1}^n S^2 \times S^2$ along any n -component link L . We denote the link surgery by A_L . Since the following diffeomorphism

$$E(1) \#_{T=T_i} S^2 \times S^2 \cong E(1) \#^2 S^2 \times S^2 = \#^3 \mathbb{C}P^2 \#^{11} \overline{\mathbb{C}P^2} \quad (3)$$

holds, we have $SW_{A_L} = 0$. The first diffeomorphism is due to Figure 9. The leftmost figure is a submanifold of $E(1) \#_{T=T_i} S^2 \times S^2$, where the handle decomposition of $E(1)$ uses the diagram of Figure 8.10 in [GS]. Sliding handles several times, we find a separated Hopf link (the rightmost figure). The Hopf link is unlinked from other attaching handles of $E(1) \#_{T=T_i} S^2 \times S^2$ while the handle sliding is operated. The second equality of (3) holds by some blow ups and downs. Thus by vanishing theorem of SW -invariant, we have $A_L = 0$.

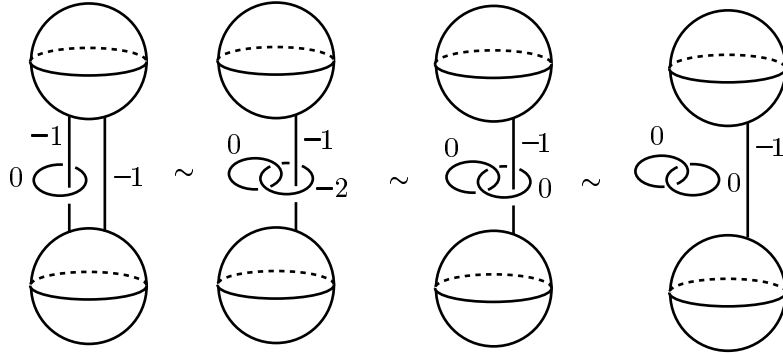


Figure 9: $E(1) \#_{T=T_i} S^2 \times S^2 = E(1) \#^2 S^2 \times S^2$

Proof of Theorem 2. Let $L = K_1 \cup K_2 \cup \dots \cup K_n$ be any n -component link. The set $\tilde{\mathcal{L}}_n$ of all n -component links up to local three strand twist consists of 2^{n-1} classes due to Nakanishi and Ohyaama's results [Oh, Na]. Forgetting the ordering of the components of any link in $\tilde{\mathcal{L}}_n$ we get a set \mathcal{L}_n . The set \mathcal{L}_n has n classes. The class is represented by $L_{n,\ell}$ ($\ell = 0, 1, \dots, n-1$) as in Figure 10.

By applying three strand twist method to link surgery A_L we have only to consider diffeomorphism type of $A_{L_{n,\ell}}$ for some ℓ .

Now suppose that $\ell \geq 1$. The ℓ and $n - \ell - 1$ components in Figure 10 are parallel each other. Then we embed $T^2 \times D^2$ in $S^3 \times S^1$ as $S_i \times S^1$ ($i = 1, 2, 3$) where $S_1, S_2,$

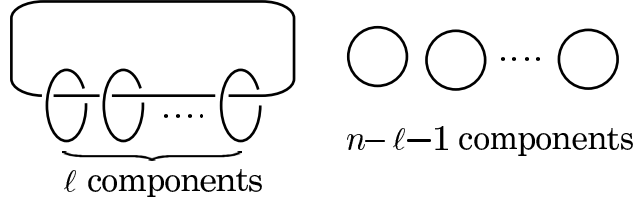


Figure 10: The representation $L_{n,\ell}$ of \mathcal{L}_n

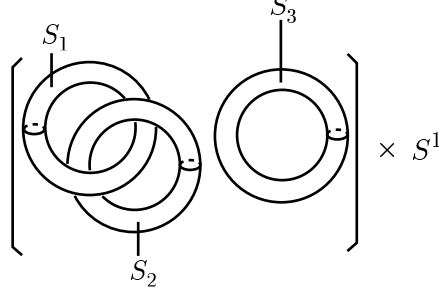


Figure 11: Torus neighborhoods $S_i \times S^1$ in $S^3 \times S^1$.

and S_3 are embedded solid tori in S^3 as in Figure 11. Let $C(m)$ be elliptic fibration over D^2 with exactly m parallel cusp fibers as singular fibers. $A_{L_{n,\ell}}$ is obtained by attaching $\overline{C(1)} \cup C(1) - \nu(T_1)$ to $\partial(\nu(S_1) \times S^1)$, $\overline{C(\ell)} \cup C(\ell) - \nu(T_2)$ to $\partial(\nu(S_2) \times S^1)$ and $\overline{C(n-\ell-1)} \cup C(n-\ell-1) - \nu(T_3)$ to $\partial(\nu(S_3) \times S^1)$. The tori T_1 , T_2 and T_3 are general fibers in the cusp neighborhoods. Thus the diagram is Figure 12.

In this section from here on in the diagram red, black, blue components stand for -1 -, 0 -, 1 -framed 2-handles respectively. Here parallel -1 framed 2-handles with 0 framed 2-handles in Figure 13 gives rise to $S^2 \times S^2$ connected sum components in such a way as Figure 13. Moreover we can take out two $S^2 \times S^2$ connected components by some handle slidings (Figure 14). The right part of Figure 14 are moved as in Figure 15. Sliding several times as in Figure 16-17 we get the right diagram. The Hopf-linked handles of the right in Figure 17 is a $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ -component. The rest of diagram in the left of Figure 17 is another $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Thus we have $\#^{2n-3} S^2 \times S^2 \#^2 \mathbb{C}P^2 \#^2 \overline{\mathbb{C}P^2} = \#^{2n-1} \mathbb{C}P^2 \#^{2n-1} \overline{\mathbb{C}P^2}$.

Suppose that $\ell = 0$. The diagram of $A_{L_{n,0}}$ is the left of Figure 18. Applying Figure 13, we get the right of Figure 18. Sliding handle as indicated in Figure 18 gives rise to the left of Figure 19. This sliding is the same as Figure 15. Canceling handles in Figure 19, we obtain $A_{L_{n,0}} \cong \#^{2n-1} S^2 \times S^2$. \square

5 Scharlemann's manifold.

From here we focus on Scharlemann's manifolds which are defined in Section 1.3.

5.1 Scharlemann's manifolds along the meridians.

In this subsection we consider Scharlemann's manifolds by the meridian γ_0 . We remark the following.

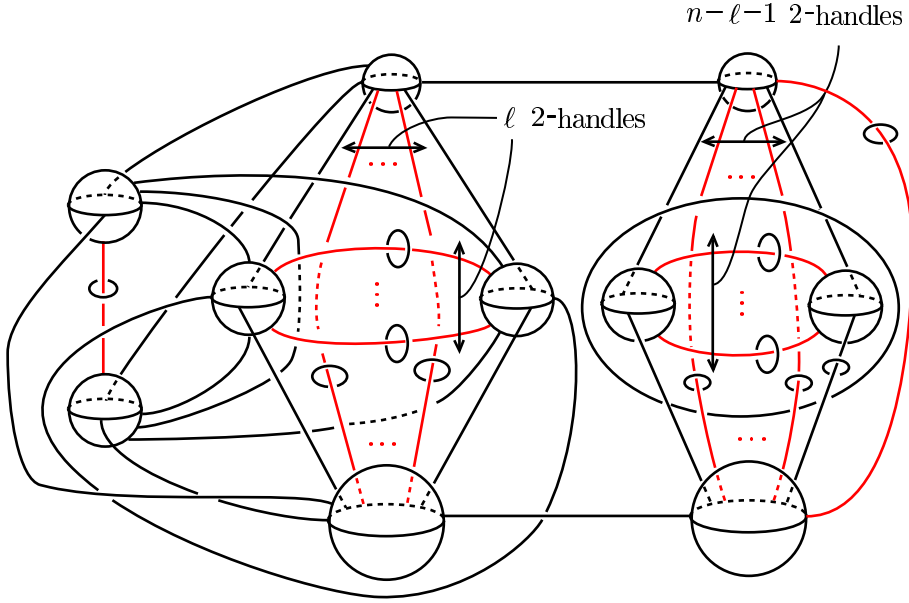


Figure 12: $\coprod_{i=1} [\overline{C} \cup C - \nu(T)] \cup [(S^3 - \nu(L_{n,\ell})) \times S^1]$.

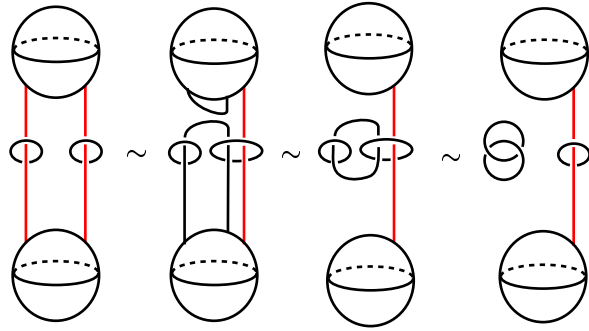


Figure 13: To make $S^2 \times S^2$ -component from two parallel -1 framed 2-handles.

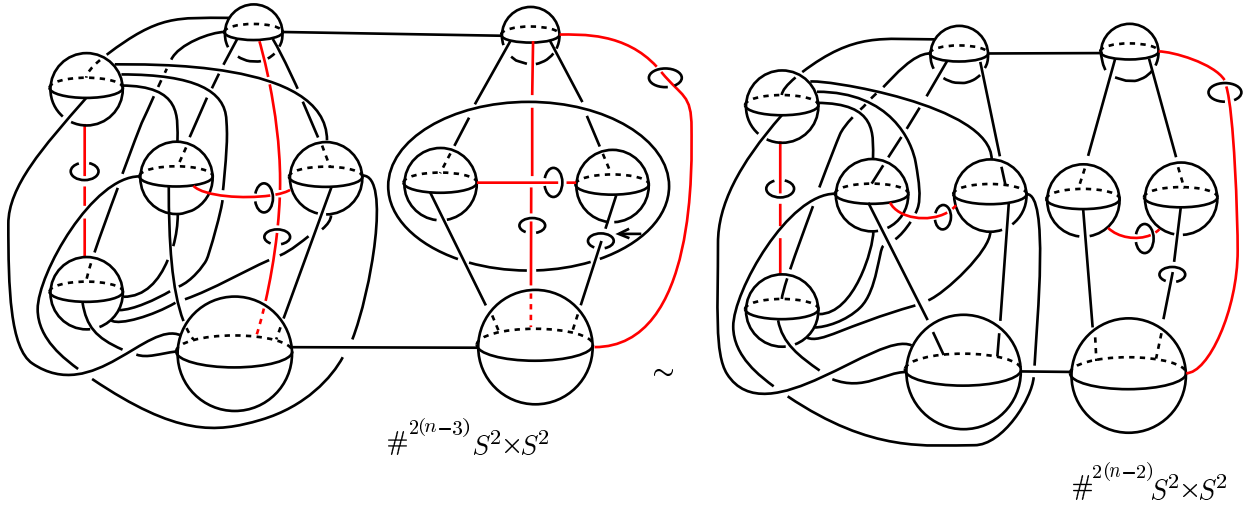


Figure 14: To come out two $S^2 \times S^2$ components.

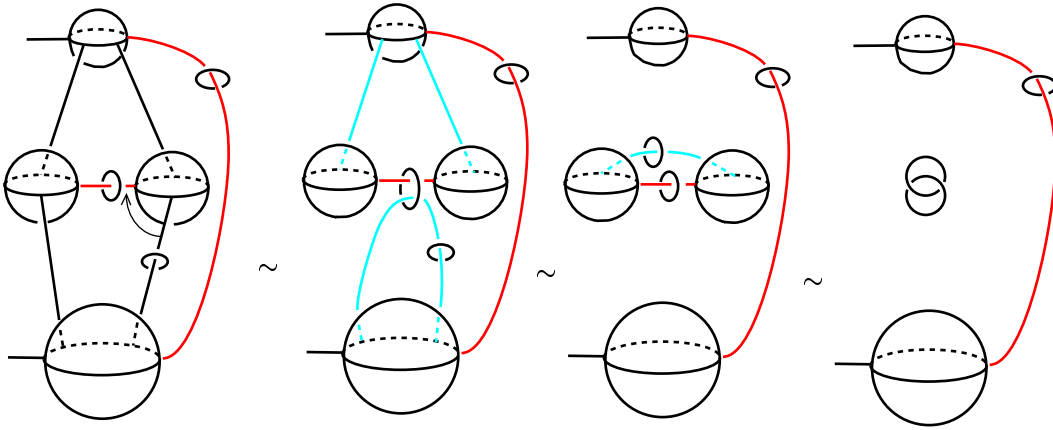


Figure 15: The right part of Figure 14.

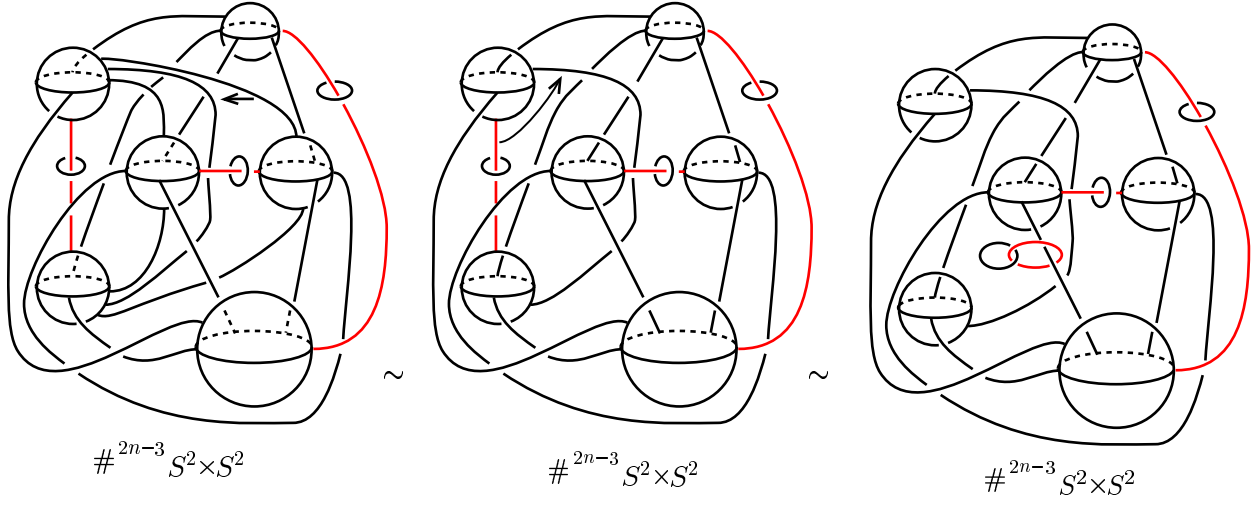


Figure 16: Several handle slidings.

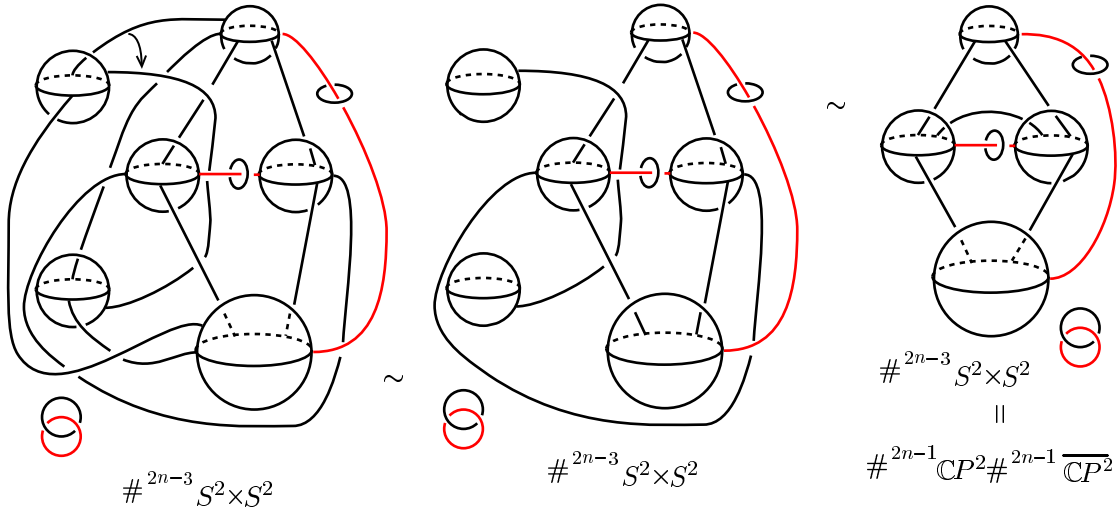


Figure 17:

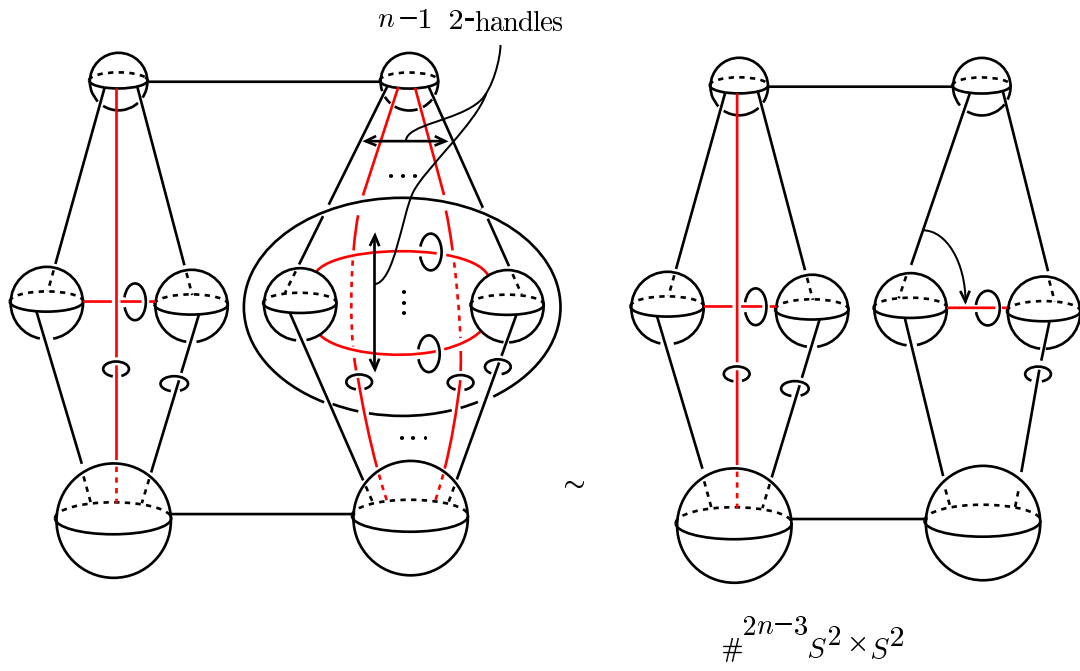


Figure 18: The $\ell = 0$ case.

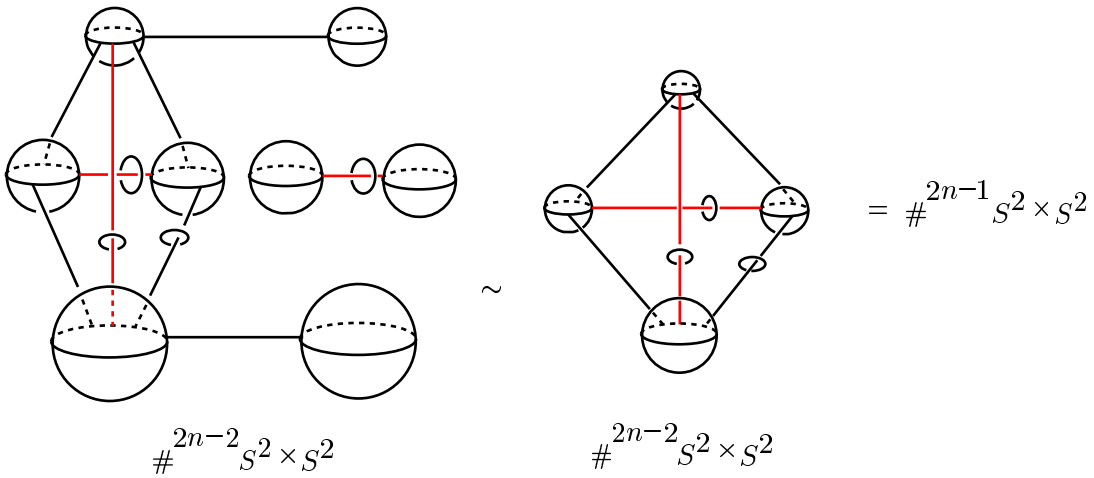


Figure 19: Proof of Theorem 2.

Remark 2. Let γ_0 be the meridian circle in $S^3_{-1}(K)$. Scharlemann's manifolds $B_K^0(\gamma_0)$ is always diffeomorphic to $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ by handle calculus.

In the case of $\epsilon = 1$, we note the relationship between $B_K^1(\gamma_0)$ and the knot surgery of the fishtail neighborhood.

Lemma 2. $B_K^1(\gamma_0)$ is diffeomorphic to $\overline{F} \cup F_K$.

The gluing map φ_0 satisfies (2) and $\varphi_0(\text{meridian of } K) = \text{the vanishing cycle}$. The resulting manifold

$$\overline{F} \cup F_K = \overline{F} \cup [F - \nu(T)] \cup_{\varphi_0} [(S^3 - \nu(K)) \times S^1]$$

is Figure 20, where the picture is the case of $K = 4_1$.

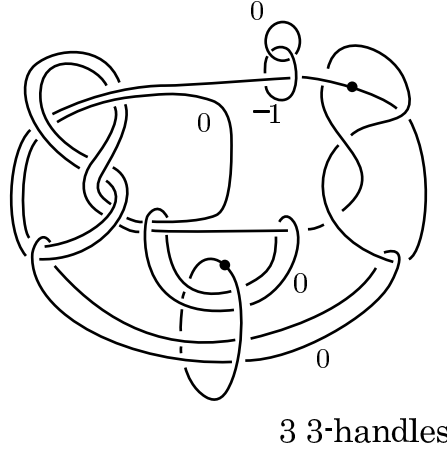


Figure 20: $\overline{F} \cup [F - \nu(T)] \cup_{\varphi_0} [(S^3 - \nu(K)) \times S^1]$.

Proof of Lemma 2. The surgery along $\gamma_0 \times \{\text{pt}\}$ in $S^3_{-1}(K) \times S^1$ is Figure 21. Hence we get the following diffeomorphisms.

$$\begin{aligned} B_K^1(\gamma_0) &= [S^3_{-1}(K) \times S^1 - \nu(\gamma_0)] \cup_{\phi'} S^2 \times D^2 = \overline{F} \cup (F - \nu(T)) \cup_{\varphi_{-1}} [S^3 - \nu(K)] \times S^1 \\ &\cong \overline{F} \cup (F - \nu(K)) \cup_{\varphi_0} S^2 \times D^2 = \overline{F} \cup F_K \end{aligned}$$

□

Proof of Theorem 4. Applying the same diffeomorphism argument as the proof of Theorem 1 and Ohya's unknotting operation imply

$$\overline{F} \cup F_{K_1} = \overline{F} \cup F \cong S^3 \times S^1 \# S^2 \times S^2.$$

The last diffeomorphism is Figure 22. □

The manifolds obtained by the same meridian surgery on $S^3_p(K) \times S^1$ are diffeomorphic to $B_K^\epsilon(\gamma_0)$ by one strand twist.

Remark 3. $B_K^1(\gamma_0)$ is obtained from A_K as a surgery along an embedded S^2 . The neighborhood of the sphere Σ is the union of the bottom 0 framed 2-handle and the 4-handle (the left of Figure 23). Attaching the 3-handle and 4-handle to the complement gets B_K^1 (the right of Figure 23). The circle δ in Figure 23 is the core circle of $S^1 \times D^3$ attached.

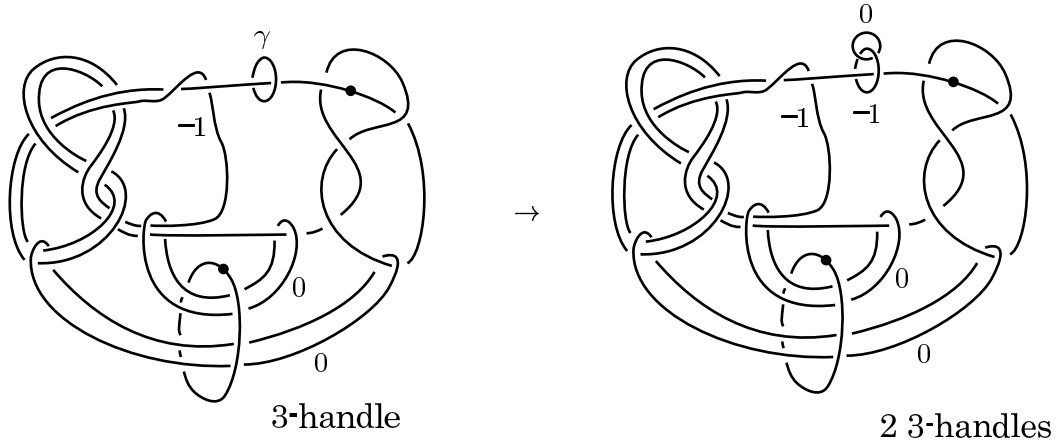


Figure 21: The non-trivial surgery along γ in $S^3_{-1}(K) \times S^1$.

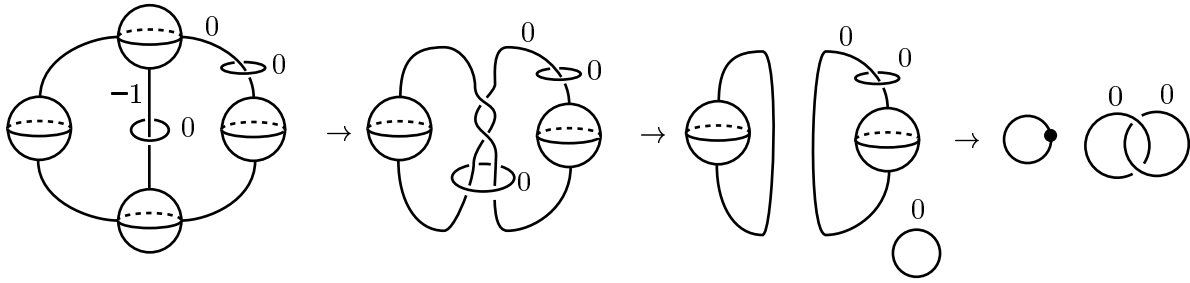


Figure 22: The double of F .

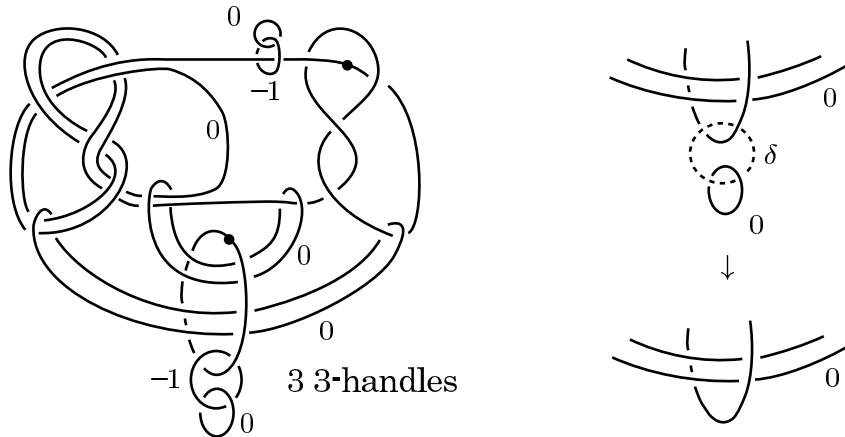


Figure 23: The left: A_K . The right: surgery $B_K^1 = [A_K - \nu(\Sigma)] \cup S^1 \times D^3$.

Remark 4. In [Ak3] Akbulut got a plug twisting $(W_{1,2}, f)$ satisfying $E(1) = N \cup_{\text{id}} W_{1,2}$ and $E(1)_{2,3} = N \cup_f W_{1,2}$. The definitions of plug, N and $W_{1,2}$ are written down in [Ak3]. In the same way as [Ak3] we can also show that there exist infinitely many plug twistings $(W_{1,2}, f_K)$ of $E(1)$ with the same plug $W_{1,2}$. As a result any plug twisting satisfies $E(1) = M \cup_{\text{id}} W_{1,2}$ and $E(1)_K = M \cup_{f_K} W_{1,2}$. Infinite variations of Alexander polynomial $\Delta_K(t)$ of knot imply the existence of infinite embeddings $W_{1,2} \hookrightarrow M \cup_{\text{id}} W_{1,2}$.

5.2 Scharlemann's manifold along a non-meridian circle.

The fundamental group of $S^3_{-1}(3_1)$ is

$$\pi_1(S^3_{-1}(3_1)) = \langle x, y | x^5 = (xy)^3 = (xyx)^2 \rangle \cong \tilde{A}_5.$$

The set of free homotopy classes of maps $S^1 \rightarrow S^3_{-1}(3_1)$ is

$$[S^1, S^3_{-1}(3_1)] = \pi_1(S^3_{-1}(3_1)) / \text{conj.} \quad (4)$$

and it possesses 9 classes as follows.

Classes	$[e]$	$[x^5]$	$[xyx]$	$[x]$	$[x^2]$	$[x^3]$	$[x^4]$	$[xy]$	$[(xy)^2]$
Order	1	2	4	10	5	10	5	6	3

Each of the classes is a normal generator of the fundamental group except $[e], [x^5]$. Let K be a knot in S^3 . For $z \in \pi_1(S^3_{-1}(3_1))$ we define $B^e_{3_1}(z)$ to be the surgery of $S^3_{-1}(3_1) \times S^1$ along a knot $K \subset S^3_{-1}(3_1)$ whose free homotopy class $[K]$ coincides with the conjugacy class $[z]$. Akbulut's result [Ak1] is that $B^1_{3_1}(x)$ is the standard $S^2 \times S^2 \# S^3 \times S^1$. For other conjugacy class we will prove the following.

Proposition 1. $B^1_{3_1}(xy)$ is diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$ and $B^0_{3_1}(xy)$ is diffeomorphic to $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

Remark 5. The conjugacy class differs from that of x , for the order of xy^{-2} is 6 while the order of x is 10. Thus the two surgeries are different in general.

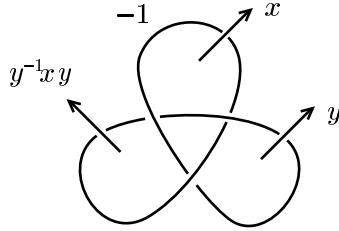


Figure 24: The generator x, y of $\pi_1(\Sigma)$.

Remark 6. Here taking the diagram in Figure 24, we fix the framing of γ .

Proof of Proposition 1. The meridian $\gamma_0 \subset S^3_1(4_1)$ is isotopic to a curve $\gamma'_0 \subset S^3_{-1}(3_1)$ as in Figure 25. The homotopy class $[\gamma'_0]$ of the curve is equal to $[xy^{-2}]$ in Equality (4). In fact $(y^{-1}xy)^2 y^{-1} = y^{-1}x^2 \sim xy^{-1}x = x^2 y^{-1} x^{-1} y^{-1} x^2 \sim x^5 (yxyx)^{-1} = yx \sim xy$ holds. Hence using one strand twist twice and Theorem 4, we get the following

diffeomorphisms

$$\begin{aligned}
B_{3_1}^1(xy) &= [S_{-1}^3(3_1) \times S^1 - \nu(\gamma'_0 \times \{\text{pt}\})] \cup_{\phi'} D^2 \times S^2 \\
&\cong [S_1^3(4_1) \times S^1 - \nu(\gamma_0 \times \{\text{pt}\})] \cup_{\phi'} D^2 \times S^2 \\
&\cong [S_{-1}^3(4_1) \times S^1 - \nu(\gamma_0 \times \{\text{pt}\})] \cup_{\phi'} D^2 \times S^2 \\
&= B_{4_1}^1(\gamma_0) \cong S^2 \times S^2 \# S^3 \times S^1.
\end{aligned}$$

The case of $B_{3_1}^0(xy)$ is diffeomorphic to $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the same way. \square

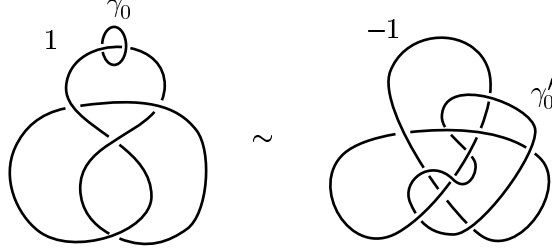


Figure 25: An isotopy of γ_0 via the diffeomorphism $S_1^3(4_1) \cong S_{-1}^3(3_1)$.

Here we will argue several other cases.

Proposition 2. $B_{3_1}^1([x^2])$, $B_{3_1}^0([x^3])$ and $B_{3_1}^1([x^4])$ are diffeomorphic to $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

Proof. Here we use notation \approx , \sim and $=$ as one strand twist, a homeomorphism, Kirby calculus technique in 3-dimansional.

In the case of $B_{3_1}^1([x^2])$, the last picture in Figure 26 represents a surgery on $L(5, -1) \times S^1$. This is diffeomorphic to $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

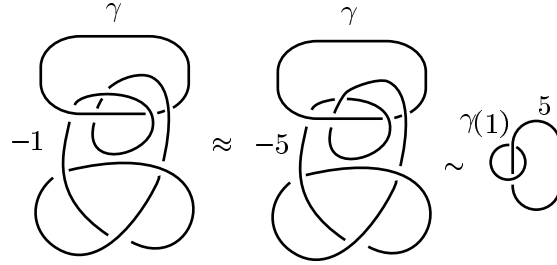


Figure 26: The diffeomorphism for $B_{3_1}^1([x^2])$.

In the case of $B_{3_1}^0([x^3])$ the last picture in Figure 27 represents a surgery along a knot γ in $S^3 \times S^1$ with odd framing. Namely the manifold is $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

In the case of $B_{3_1}^1([x^4])$, the last picture in Figure 28 gives $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the similar way. Here $Pr(-2, 3, 7)$ is the $(-2, 3, 7)$ -pretzel knot. \square

Proposition 3. $B_{3_1}^0(xy)$ is diffeomorphic to $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

Proof. The homotopy class of the curve γ is $xyy^{-1}xy \sim x^2y \sim yxx$. then from the Figure 29, we get $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. \square

Proposition 4. $B_{3_1}^1((xy)^2)$ is diffeomorphic to $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

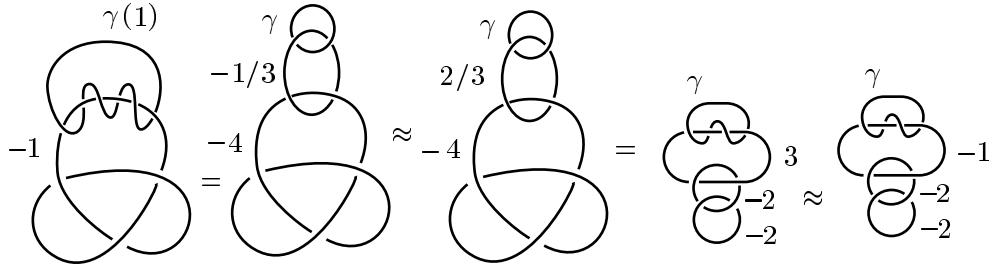


Figure 27: The diffeomorphism for $B_{3_1}^0([x^3])$.

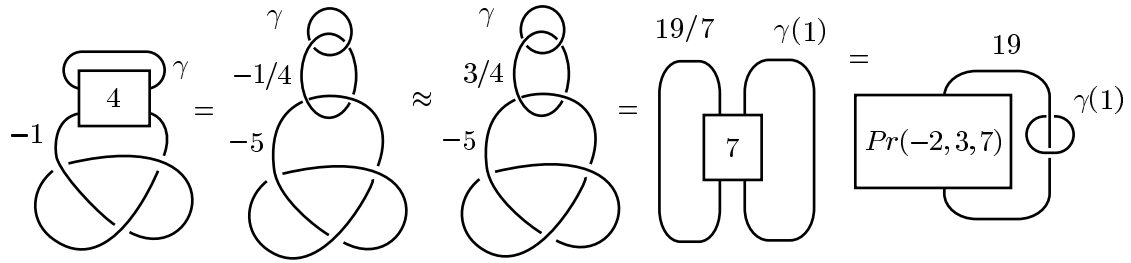


Figure 28: The diffeomorphism for $B_{3_1}^1([x^4])$.

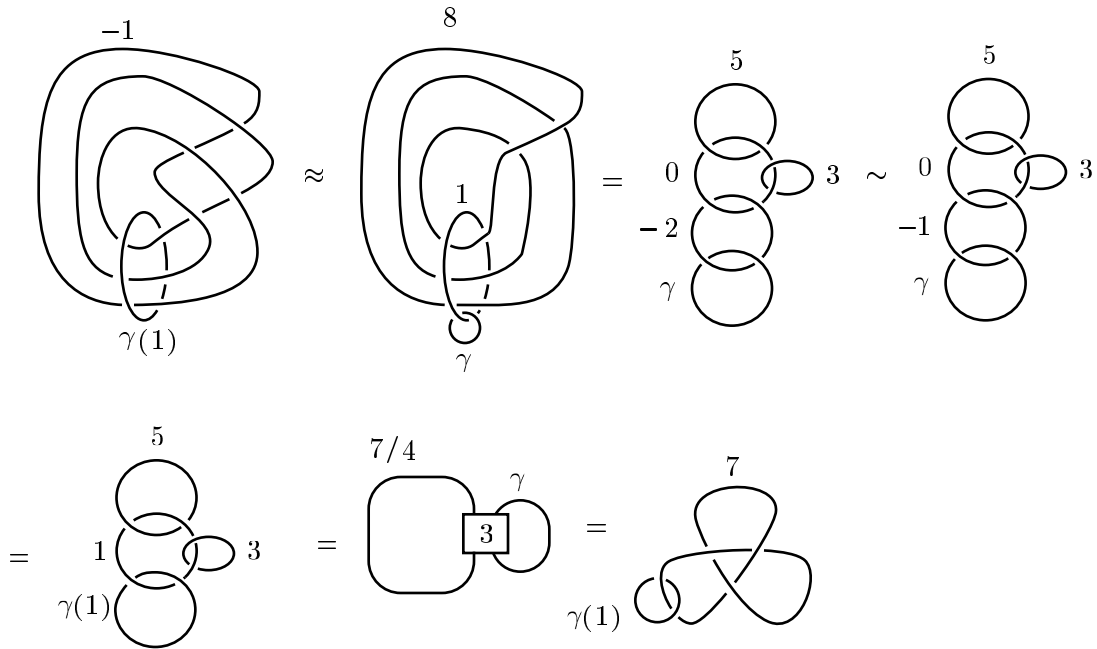


Figure 29: The diffeomorphism for $B_{3_1}^0(xy x)$.

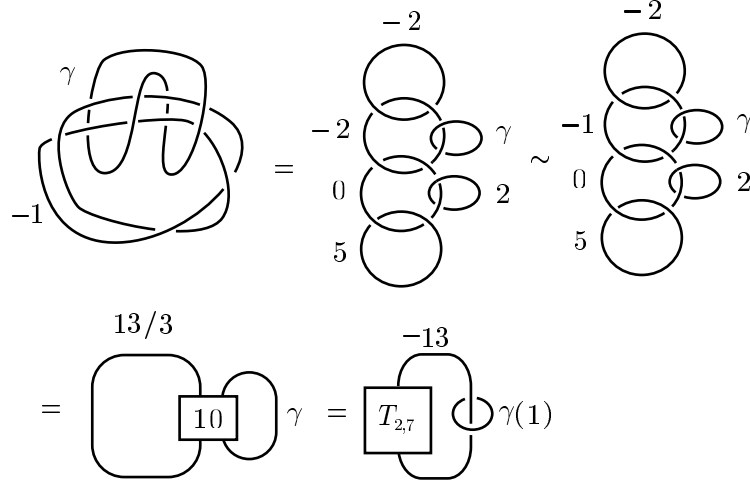


Figure 30: The diffeomorphism for $B_{31}^1((xy)^2)$.

Proof. The deformation as in Figure 30 we get $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Here $T_{2,7}$ is the $(2, 7)$ -torus knot \square .

In the end we raise the questions.

Question 1. *Are the following manifolds*

$$B_{31}^0(x^2), B_{31}^1(x^3), B_{31}^0(x^4), B_{31}^1(xy), B_{31}^0((xy)^2)$$

diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$?

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